



Control with transverse functions and a single generator of underactuated mechanical systems

Pascal Morin, Claude Samson

► To cite this version:

Pascal Morin, Claude Samson. Control with transverse functions and a single generator of underactuated mechanical systems. [Research Report] RR-5847, INRIA. 2006, pp.20. inria-00070179

HAL Id: inria-00070179

<https://inria.hal.science/inria-00070179>

Submitted on 19 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

***Control with transverse functions and a single
generator of underactuated mechanical systems***

Pascal Morin — Claude Samson

N° 5847

February 2006

_____ Thème NUM _____



***rapport
de recherche***

Control with transverse functions and a single generator of underactuated mechanical systems

Pascal Morin , Claude Samson

Thème NUM — Systèmes numériques
Projets Icare

Rapport de recherche n° 5847 — February 2006 — 20 pages

Abstract: The control of a class of underactuated mechanical systems on Lie groups is addressed, with the objective of stabilizing, in a practical sense, *any* (possibly non-admissible) reference trajectory in the configuration space. The present control design method extends a previous result by the authors to systems underactuated by more than one control. For example, it allows to address the case of a 3D-rigid body immersed in a perfect fluid with only three control inputs. The choice of the control parameters is also discussed in relation to the system's zero-dynamics.

Key-words: mechanical system, trajectory stabilization, transverse function, underactuated system

Commande de systèmes mécaniques sous-actionnés avec les fonctions transverses et un seul générateur

Résumé : Nous proposons une méthode de synthèse de retours d'état pour une classe de systèmes mécaniques sous-actionnés invariants par rapport à une opération de groupe (de Lie) définie sur l'espace de configuration. L'objectif de commande est de stabiliser, dans un sens pratique, toute trajectoire dans l'espace de configuration, admissible ou non. La méthode de synthèse de retours d'état proposée étend un résultat que nous avons obtenu précédemment à des systèmes dont le degré de sous-actionnement est supérieur à un. Elle s'applique par exemple à un corps rigide 3-D immergé dans un fluide parfait, dans le cas où on ne dispose que de trois entrées de commande. Le choix de certains paramètres de commande, basé sur une étude de zéro-dynamique, est aussi abordé.

Mots-clés : fonction transverse, stabilisation de trajectoire, système mécanique, système sous-actionné

1 Introduction

This paper addresses the control of underactuated (mechanical) systems the dynamics of which can be modeled in the form

$$\begin{cases} \dot{g} = X(g)\xi := \sum_{i=1}^n X_i(g)\xi_i \\ \dot{\xi} = Q(\xi) + \sum_{i=1}^m b_i u_i \quad (m < n) \end{cases} \quad (1)$$

with g the system's configuration (e.g. position and orientation) belonging to an n -dimensional connected Lie group G , $\{X_1, \dots, X_n\}$ a *left-invariant* basis of the group's Lie algebra \mathfrak{g} , $\xi \in \mathbb{R}^n$ a vector of *instantaneous velocities*, Q a quadratic vector-valued function containing the terms associated with Coriolis and centrifugal forces, $\{b_1, \dots, b_m\}$ independent vectors of \mathbb{R}^n , and $u = (u_1, \dots, u_m)$ the vector of control inputs produced by the actuators. Such a system is *invariant* on the Lie group G in the sense that, given an initial velocity $\xi(0)$ then, whatever the input function $t \mapsto u(t)$ ($t \geq 0$) applied to the system, the associated trajectory originated at some point g_1 is the same as the one originated at another point g_2 , modulo a *fixed* translation on the group. The fact that $m < n$ (by assumption) makes the control of this class of systems particularly challenging. In this respect, note that the linearization of System (1) at any fixed point $(g, \xi) = (g, 0)$ is not controllable, and also that Brockett's necessary condition [3] for the existence of smooth state feedbacks that asymptotically stabilize a fixed configuration is not satisfied in this case.

The difficulties associated with the control of this class of underactuated mechanical systems have motivated many studies in recent years, and a variety of problems have been addressed: characterization of the controllability properties [11], of the asymptotic stabilizability of fixed points [6], open-loop control design [4], asymptotic stabilization of fixed configurations with different types of feedbacks [18, 7, 15, 4, 13], asymptotic stabilization of *specific* non-constant admissible trajectories [5, 10, 1]. Recently, in [17], we have proposed a control design method for the stabilization of *general* reference trajectories $g_r(t)$ ($t \geq 0$), i.e. arbitrary smooth curves on G which are not necessarily solutions to System (1) for some control input $u_r(t)$. This control method relies on the transverse function approach, initially developed by the authors for the control of nonholonomic systems [16]. The feedback laws derived in [17] are *practical stabilizers*, i.e. they ensure a bounded tracking error, the ultimate norm of which can be upper-bounded by any (strictly positive) pre-specified value, via the choice of the control parameters. Note that, for non-admissible reference trajectories, *asymptotic* stabilization cannot be achieved anyway. Moreover, even by restricting the set of reference trajectories to admissible ones, the generic problem of asymptotic stabilization remains ill-posed [12].

In this paper, we develop the control approach proposed in [17] further. First, we extend the control design method to a larger class of systems. More precisely, in [17], we addressed the case of mechanical systems underactuated by one control only (i.e. $m = n - 1$) and we assumed a specific structure for the term $Q(\xi)$ in (1). The solution here proposed applies

to a much larger class of systems. For example we show that the very challenging example of a rigid body on $SE(3)$ with three control forces (see e.g. [4]), for which $m = 3 = n - 3$, can be treated with our approach. For Lie groups of dimension three, we also show that the present approach applies provided only that the system is STLC (i.e. Small Time Locally Controllable). Another issue, which was not addressed in [17], concerns the selection of the generator used in the control design in relation to the asymptotical behavior of the controlled system's solutions. This involves the study of the system's "zero-dynamics" in the special case where the reference trajectory is a fixed point in the configuration space. A general treatment of this issue is not available yet, and only a case study is presented here.

Let us finally mention two recently published papers related to the present approach. In [2], the control of an underactuated surface vessel is addressed, also with an objective of practical stabilization, but the concept of transverse function is absent (a notion of a dynamic oscillator is used instead), the properties of systems on Lie groups are not explicitly exploited, and only the case of admissible trajectories is considered. In [14], another control design strategy, also based on the transverse function approach, is proposed for a large class of underactuated systems. It yields a different asymptotical behavior of the closed-loop system's solutions (compare in particular [14, Sec. VI] and Section 6 of the present paper). A more thorough comparison of the two ways of using transverse functions for underactuated mechanical systems still has to be conducted.

2 Notation and recalls

2.1 Special vectors and matrices

Throughout the paper, the transpose of a vector x in \mathbb{R}^n is denoted as x' . The i -th vector of the canonical basis of \mathbb{R}^n is denoted as e_i , i.e. all components of e_i are equal to zero except for the i -th one which is equal to one. The cross product in \mathbb{R}^3 is denoted as \times and \hat{x} is the skew-symmetric matrix associated with this product, i.e. $\hat{x}y = x \times y$. With $x = (x'_1, x'_2)'$ a vector in \mathbb{R}^6 such that $x_1 \in \mathbb{R}^3$ and $x_2 \in \mathbb{R}^3$ we associate the matrix $\hat{x} = \begin{pmatrix} \hat{x}_1 & x_2 \\ 0_{3 \times 3} & 0_3 \end{pmatrix}$.

2.2 Systems on Lie groups

Let G denote a connected Lie group of dimension n . The unit element of G is denoted as e , i.e. $\forall g \in G : ge = eg = g$. The inverse g^{-1} of $g \in G$ is the (unique) element in G such that $gg^{-1} = g^{-1}g = e$. The left (resp. right) translation operator on G is denoted as l (resp. r), i.e. $\forall(\sigma, \tau) \in G^2 : l_\sigma(\tau) = r_\tau(\sigma) = \sigma\tau$. A v.f. X on G is left-invariant iff $\forall(\sigma, \tau) \in G^2, dl_\sigma(\tau)X(\tau) = X(\sigma\tau)$, with df denoting the differential of the function f . The Lie algebra –of left-invariant v.f.– of the group G is denoted as \mathfrak{g} . The adjoint representation of G is denoted as Ad , i.e. $\forall\sigma \in G, \text{Ad}(\sigma) := dI_\sigma(e)$, with $I_\sigma : G \rightarrow G$ defined by $I_\sigma(g) := \sigma g \sigma^{-1}$. By extension of the definition of Ad , we define $\text{Ad}(\sigma)X(g) := dl_g(e)\text{Ad}(\sigma)X(e)$. If $X \in \mathfrak{g}$, $\exp(tX)$ is the solution at time t of $\dot{g} = X(g)$

with the initial condition $g(0) = e$. A driftless control system $\dot{g} = \sum_{i=1}^m X_i(g)\xi_i$ on G is said to be left-invariant on G if the control v.f. X_i are left-invariant. Given a family $Y := \{Y_1, \dots, Y_p\}$ of vector fields on G and a vector $\xi \in \mathbb{R}^p$, we denote by $Y(g)\xi$ the vector field $\sum_{i=1}^p Y_i(g)\xi_i$ (this notation is already used in Eq. (1)).

Let $X = \{X_1, \dots, X_n\}$ denote a basis of \mathfrak{g} . If $(g_a(t), \xi_a(t))$ and $(g_b(t), \xi_b(t))$ ($t \geq 0$) are two solutions to $\dot{g} = X(g)\xi$, then (by omitting the time index)

$$\frac{d}{dt}(g_a g_b^{-1}) = X(g_a g_b^{-1}) \text{Ad}^X(g_b)(\xi_a - \xi_b) \quad (2)$$

with Ad^X the (invertible) matrix-valued function defined by $\forall \sigma \in G, \forall \xi \in \mathbb{R}^n, \text{Ad}(\sigma)X(e)\xi = X(e)\text{Ad}^X(\sigma)\xi$. According to this definition, $\text{Ad}^X(e) = I_n$, with I_n the identity matrix associated with \mathbb{R}^n . We have also

$$\frac{d}{dt}(g_a^{-1} g_b) = X(g_a^{-1} g_b)(\xi_b - \text{Ad}^X(g_b^{-1} g_a)\xi_a) \quad (3)$$

Let $d_G : (g_a, g_b) \mapsto d_G(g_a, g_b)$ denote a distance on G , left-invariant w.r.t. the group operation, i.e. such that $\forall g_a, b, c \in G, d_G(g_b, g_c) = d_G(g_a g_b, g_a g_c)$. Then, for any $\gamma \geq 0$, we denote by $B_G(\gamma) := \{g \in G : d_G(g, e) \leq \gamma\}$ the closed ball of radius γ centered at e .

2.3 Transverse Functions

Let

- _ \mathbb{T}^k denote the k -dimensional torus, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,
- _ $X = \{X_1, \dots, X_n\}$ denote a basis of \mathfrak{g} ,
- _ f denote a smooth function from \mathbb{T}^{n-m} ($m < n$) to a neighborhood $\mathcal{U} \subset G$ of e .

Then, there exists a matrix-valued function C such that, along any differentiable path $\alpha(t)$ on \mathbb{T}^{n-m} , one has

$$\begin{aligned} \dot{f}(\alpha) &= X(f(\alpha))C(\alpha)\dot{\alpha} \\ &= X^1(f(\alpha))C^1(\alpha)\dot{\alpha} + X^2(f(\alpha))C^2(\alpha)\dot{\alpha} \end{aligned} \quad (4)$$

with $X^1 = \{X_1, \dots, X_m\}$ and $X^2 = \{X_{m+1}, \dots, X_n\}$. The function f is said to be *transversal to the v.f. X_1, \dots, X_m* iff $C^2(\alpha)$ is invertible $\forall \alpha \in \mathbb{T}^{n-m}$. The transverse function theorem given in [16] asserts the existence of such functions, whatever the size of \mathcal{U} , provided that the Lie algebra generated by the family X^1 is equal to \mathfrak{g} . It also provides a general expression for a family of such functions.

3 Asymptotic stabilization in the full-actuation case

Before presenting a solution for the stabilization of trajectories in the case of underactuated systems, let us first recall a control solution to this problem in the simpler case of fully actuated systems. The system's equations are then given by

$$\begin{cases} \dot{g} &= X(g)\xi \\ \dot{\xi} &= u \end{cases} \quad (5)$$

Consider a trajectory of reference configurations $g_r(t)$, and denote by $\xi_r(t)$ the associated velocity vector (assumed differentiable), i.e. $\forall t > 0, \dot{g}_r(t) = X(g_r(t))\xi_r(t)$. The element $\tilde{g}(t) := g_r(t)^{-1}g(t)$ characterizes the tracking error at time t . By using (3) one obtains the following *error system*:

$$\begin{cases} \dot{\tilde{g}} &= X(\tilde{g})(\xi - \text{Ad}^X(\tilde{g}^{-1})\xi_r) \\ \dot{\xi} &= u \end{cases} \quad (6)$$

and $(\tilde{g}, \xi) = (e, \xi_r)$ is a solution to this control system, associated with the control input $u = \xi_r$. The control problem consists in stabilizing this solution. Let V denote a twice differentiable positive function on G , such that for some constants $\gamma, \alpha_m, \alpha_M, \beta_m, \beta_M > 0$, and for any $g \in B_G(\gamma)$,

$$\begin{aligned} \text{P1} : & \alpha_m d_G^2(g, e) \leq V(g) \leq \alpha_M d_G^2(g, e) \\ \text{P2} : & \beta_m V(g) \leq \sum_{i=1}^n (dV(g)X_i(g))^2 \leq \beta_M V(g) \end{aligned} \quad (7)$$

Such a function always exists, for instance in the form of a quadratic function when working with a system of coordinates.

Proposition 1 *Let*

$$\begin{aligned} u := & -k(\xi - \text{Ad}^X(\tilde{g}^{-1})\xi_r - \xi^*(\tilde{g})) + \text{Ad}^X(\tilde{g}^{-1})\dot{\xi}_r \\ & + d(F_{\xi_r} + \xi^*)(\tilde{g}) \left(X(\tilde{g})(\xi - \text{Ad}^X(\tilde{g}^{-1})\xi_r) \right) \end{aligned} \quad (8)$$

with $k > 0$, $F_{\xi_r}(\tilde{g}) := \text{Ad}^X(\tilde{g}^{-1})\xi_r$, and

$$\xi_i^*(\tilde{g}) := -k_i dV(\tilde{g})X_i(\tilde{g}) \quad (k_i > 0; i = 1, \dots, n) \quad (9)$$

Then, the feedback control (8) applied to the system (6) exponentially stabilizes the solution $(\tilde{g}, \xi) = (e, \xi_r)$.

The proof consists in verifying that $\tilde{V}(\tilde{g}, \tilde{\nu}) := V(\tilde{g}) + \mu \|\tilde{\nu}\|^2$, with $\mu > 0$ large enough and

$$\tilde{\nu} := \xi - \text{Ad}^X(\tilde{g}^{-1})\xi_r - \xi^*(\tilde{g}) \quad (10)$$

is a Lyapunov function for the controlled system (see [17] for more details).

4 Practical stabilization for a class of underactuated systems

Let us assume that for some basis $X = \{X_1, \dots, X_n\}$ of \mathfrak{g} , and by a proper choice of both the velocity variable ξ and the control variable u , System (1) can be written as:

$$\begin{cases} \dot{g} &= X(g)\xi \\ \dot{\xi}_1 &= u_1 \\ \dot{\xi}_2 &= \xi_1 A \xi_2 + B u_2 + P(\xi_2) \end{cases} \quad (11)$$

with $\xi = (\xi_1, \xi_2)'$, $\xi_1, u_1 \in \mathbb{R}$, $\xi_2 \in \mathbb{R}^{n-1}$, $u_2 \in \mathbb{R}^{m-1}$, P a quadratic function of ξ_2 , and A, B matrices such that:

Assumption 1 *The pair (A, B) is controllable, i.e. $\text{Rang}(B, AB, \dots, A^{n-2}B) = n - 1$.*

One of the reasons for studying this class of systems is stated in the following proposition, the proof of which is given in the appendix.

Proposition 2 *If Assumption 1 is satisfied, then System (11) (and thus System (1)) is STLC (i.e. Small Time Locally Controllable) at any equilibrium point $(g, \xi) = (g_0, 0)$.*

Furthermore, for Lie groups of dimension three, a stronger result can be stated:

Proposition 3 *When $n = \dim(G) = 3$ System (1) is STLC at any equilibrium point $(g, \xi) = (g_0, 0)$ if and only if it can be written as System (11) with Assumption 1 being satisfied.*

Proposition 3, which is also proved in the appendix, shows that System (11) is a generic model, in dimension three, for STLC underactuated systems whose drift term $Q(\xi)$ is quadratic. Many examples previously studied in the literature belong to this class, like e.g. the 3-d second-order chained systems on \mathbb{R}^3 , the underactuated planar PPR manipulator and the planar rigid body (hovercraft) on $SE(2)$, and the underactuated satellite with two thruster control torques on $SO(3)$ (see [17] for more details).

In [17], we have proposed a control design method for the trajectory stabilization problem here considered, in the specific case of Lie groups of dimension three, and when

$$A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P = 0 \quad (12)$$

with a some non-zero constant. Let us remark that, even in dimension three, there exist STLC underactuated mechanical systems which do not belong to this sub-class. Such is the case, for example, of the satellite with two control torques on $SO(3)$ when the torque axes are not aligned with principal axes of inertia. We show below that the control design proposed in [17] can be extended to all systems of the form (11) that satisfy Assumption 1.

Remark: The particular role played by the one-dimensional variable ξ_1 in (11) will be used in the forthcoming control design. Such a variable, which is not unique in general, is here called a “generator”.

With the notation of Section 3, the error system associated with (11) is

$$\begin{cases} \dot{\tilde{g}} &= X(\tilde{g})(\xi - \text{Ad}^X(\tilde{g}^{-1})\xi_r) \\ \dot{\xi}_1 &= u_1 \\ \dot{\xi}_2 &= \xi_1 A \xi_2 + B u_2 + P(\xi_2) \end{cases} \quad (13)$$

and the problem consists in determining a feedback control law which (practically) stabilizes the point $\tilde{g} = e$ for this system. To this end the following lemma will be used.

Lemma 1 *Let Y_1, \dots, Y_m denote the vector fields on \mathbb{R}^n defined by*

$$Y_1(y) = \begin{pmatrix} 1 \\ A y_2 \end{pmatrix}, \quad Y_i(y) = \begin{pmatrix} 0 \\ B e_i \end{pmatrix} \quad (i = 2, \dots, m) \quad (14)$$

with $y = (y_1, y_2)'$, $y_2 \in \mathbb{R}^{n-1}$, and e_i the i -th unit vector in \mathbb{R}^{n-1} . Then,

i) \mathbb{R}^n , endowed with the composition law \circ defined by

$$y \circ \bar{y} = \begin{pmatrix} y_1 + \bar{y}_1 \\ e^{A \bar{y}_1} y_2 + \bar{y}_2 \end{pmatrix} \quad (15)$$

is a Lie group, denoted as H , and Y_1, \dots, Y_m are left-invariant vector fields on H ,

ii) if Assumption 1 is satisfied, then the Lie algebra generated by $\{Y_1, \dots, Y_m\}$ coincides with the Lie algebra \mathfrak{h} of H , and there exist vector fields Y_j , ($j = m+1, \dots, n$), of the form $Y_j = \text{ad}_{Y_1}^k(Y_i)$ with $i \in \{1, \dots, m\}$, such that $\{Y_1, \dots, Y_n\}$ is a basis of \mathfrak{h} .

The proof is straightforward and left as an exercise to the willing reader.

It follows from this lemma, by application of the result recalled in Section 2.3, that there exist functions $f : \mathbb{T}^{n-m} \rightarrow \mathcal{U}$, with \mathcal{U} an arbitrary small neighborhood of the origin in \mathbb{R}^n , which are transversal to the v.f. Y_1, \dots, Y_m . Let us decompose f as $f = (f_1, f_2)'$ with f_1 (resp. f_2) a \mathbb{R} -valued (resp. \mathbb{R}^{n-1} -valued) function. One can verify that the condition of transversality is equivalent to the invertibility of the $(n-1) \times (n-1)$ matrix

$$D(\alpha) := \begin{pmatrix} B & -(\frac{\partial f_2}{\partial \alpha} - A f_2 \frac{\partial f_1}{\partial \alpha})(\alpha) \end{pmatrix} \quad (16)$$

for any $\alpha \in \mathbb{T}^{n-m}$.

Define¹

$$z_2 := e^{-A f_1(\alpha)}(\xi_2 - f_2(\alpha)) \quad (17)$$

¹Note that z_2 corresponds to the second component of $y \circ f(\alpha)^{-1}$ with $y = (y_1, \xi_2)'$.

For any smooth time-function $\alpha(\cdot)$, the time-derivative of z_2 along the solutions of System (13) is given by

$$\dot{z}_2 = e^{-Af_1(\alpha)} (\bar{\xi}_1 A \xi_2 + D(\alpha) \begin{pmatrix} u_2 \\ \alpha \end{pmatrix} + P(\xi_2)) \quad (18)$$

with

$$\bar{\xi}_1 := \xi_1 - \dot{f}_1(\alpha) \quad (19)$$

Note how the derivative of the variable α on which the transverse function f depends appears as an extra control variable in (18).

Define

$$\bar{g} := \tilde{g} h_1(\alpha)^{-1} \quad \text{with} \quad h_1(\alpha) := \exp(f_1(\alpha) X_1) \quad (20)$$

Since $\dot{h}_1(\alpha) = X_1(h_1(\alpha)) \dot{f}_1(\alpha)$, it follows from (2), (13), and (19), that

$$\dot{\bar{g}} = X(\bar{g}) \text{Ad}^X(h_1(\alpha)) (\bar{\xi} - \text{Ad}^X(\bar{g}^{-1}) \xi_r) \quad (21)$$

with $\bar{\xi} = (\bar{\xi}_1, \xi_2')'$. Finally, define (compare with (10))

$$\bar{\nu} := \begin{pmatrix} \bar{\xi}_1 \\ z_2 \end{pmatrix} - \text{Ad}^X(\bar{g}^{-1}) \xi_r - \xi^*(\bar{g}) \quad (22)$$

with $\xi^*(\bar{g})$ denoting any smooth feedback law which exponentially stabilizes the point $\bar{g} = e$ for the system $\dot{\bar{g}} = X(\bar{g}) \bar{\xi}$. An example of such a feedback is given by relation (9) in Proposition 1. It follows from (17) that

$$\bar{\xi} = T(f_1(\alpha))(\bar{\nu} + \text{Ad}^X(\bar{g}^{-1}) \xi_r + \xi^*(\bar{g})) + \begin{pmatrix} 0 \\ f_2(\alpha) \end{pmatrix}$$

with

$$T(f_1) = \begin{pmatrix} 1 & 0 \\ 0 & e^{Af_1} \end{pmatrix}$$

From there, the idea for the control design is as follows. When $\bar{\nu}$ is close to zero and the transverse function f is “small” (i.e. $f(\mathbb{T}^{n-m})$ is included a small neighborhood of e), it follows from the above expressions that $\bar{\xi}$ is close to $\text{Ad}^X(\bar{g}^{-1}) \xi_r + \xi^*(\bar{g})$ and, from (20), that $h_1(\alpha)$ is close to e whatever α . Therefore, System (21) behaves approximately like the system given by $\dot{\bar{g}} = X(\bar{g})(\bar{\xi} - \text{Ad}^X(\bar{g}^{-1}) \xi_r)$ or, in view of the abovementioned approximation of $\bar{\xi}$, like the system $\dot{\bar{g}} \approx X(\bar{g}) \xi^*(\bar{g})$, with $\bar{g}(t)$ (locally) converging to zero exponentially. This in turn yields, from (20), the convergence of \bar{g} to $h_1(\mathbb{T}^{n-m})$, and thus the ultimate boundedness of the tracking error, with a tracking precision directly related to the size of f . In order to justify this control strategy more rigorously, one must *i*) design control laws that ensure the convergence of $\bar{\nu}$ to zero, and *ii*) prove the ultimate boundedness of the tracking error when using these control laws.

Stabilization of $\bar{\nu} = 0$: By using (18)–(22), one verifies that the derivative of $\bar{\nu}$ along the solutions to System (13) is given by

$$\begin{cases} \dot{\bar{\nu}}_1 = u_1 - \ddot{f}_1(\alpha) + r_1 + \bar{\nu}_1 s_1 \\ \dot{\bar{\nu}}_2 = e^{-Af_1(\alpha)} D(\alpha) \begin{pmatrix} u_2 \\ \alpha \end{pmatrix} + r_2 + \bar{\nu}_1 s_2 \end{cases}$$

with r_i, s_i ($i = 1, 2$) denoting some functions which depend on $\bar{g}, \bar{v}_2, \alpha, \xi_r$, and $\dot{\xi}_r$, but not on \bar{v}_1 . From there it is not difficult to derive (dynamic) feedback laws that make $\bar{v} = 0$ exponentially stable:

Lemma 2 *Consider the smooth feedback control defined by*

$$\begin{cases} \begin{pmatrix} u_2 \\ \dot{\alpha} \end{pmatrix} &= D(\alpha)^{-1} e^{A f_1(\alpha)} (-k \bar{v}_2 - r_2) \\ u_1 &= \dot{\alpha}' \frac{\partial^2 f_1}{\partial \alpha^2}(\alpha) \dot{\alpha} + \frac{\partial f_1}{\partial \alpha}(\alpha) \alpha^{(2)} - k \bar{v}_1 - r_1 - \bar{v}_1 (s_1 + \bar{v}_2' s_2) \end{cases} \quad (23)$$

with $k > 0$, $\alpha(0)$ equal to any value, and $\alpha^{(2)}$ the function depending on $\bar{g}, \bar{v}, \alpha, \xi_r, \dot{\xi}_r$, and $\ddot{\xi}_r$, whose value coincides with the time-derivative of the control input $\dot{\alpha}$ along the controlled system. Then, along the trajectories of the controlled system (13)-(23),

$$\frac{1}{2} \frac{d}{dt} \|\bar{v}\|^2 = -k \|\bar{v}\|^2$$

so that $\bar{v} = 0$ is exponentially stable.

The proof is straightforward.

Ultimate boundedness of the tracking error: The following proposition, the proof of which closely follows the proof of [17, Prop. 2], establishes the ultimate boundedness of the tracking error associated with the feedback law (23).

Proposition 4 *Choose $\alpha(0)$ such that $\frac{\partial f_1}{\partial \alpha}(\alpha(0)) = 0$, and let η denote a class- \mathcal{K} function such that $\max_{\alpha} (\|f(\alpha)\| + d_G(h_1(\alpha), e) + \|I_3 - \text{Ad}^X(h_1(\alpha))\|) \leq \eta(\varepsilon)$ with $\varepsilon := \max_{\alpha} (\|f(\alpha)\|)$. Then, for any constant K_r , there exists $\varepsilon_0, \gamma_g, \gamma_v, \beta > 0$ such that, for any reference trajectory g_r such that $\|\xi_r\| \leq K_r$, and for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\left. \begin{aligned} d_G(\tilde{g}(0), e) &\leq \gamma_g \\ \|(\xi - \xi_r)(0)\| &\leq \gamma_v \end{aligned} \right\} \Rightarrow d_G(\tilde{g}, e) \text{ is u.b. by } \beta \eta(\varepsilon) \quad (24)$$

where “u.b.” means “ultimately bounded”. Moreover, if $\|\dot{\xi}_r(t)\|$ and $\|\ddot{\xi}_r(t)\|$ are bounded, then $\|\xi(t)\|$ and the control inputs $u_1(t)$, $u_2(t)$, and $\dot{\alpha}(t)$, are bounded.

The important points of this proposition are *i)* the existence of an ultimate bound for the closed-loop tracking error, *ii)* the (theoretical) possibility of reducing this bound as much as desired by choosing a “small” enough transverse function, and *iii)* the insurance that the attraction domain contains an open set whose size depends neither on the reference trajectory (given an upperbound of $\|\xi_r\|$) nor on $\varepsilon \in (0, \varepsilon_0]$.

5 Example: the rigid body in $SE(3)$

By Proposition 3 the proposed control design approach applies to any STLC system on a Lie group of dimension three. In order to illustrate its application to systems of higher dimensions we consider here the example of a rigid body on $SE(3)$ immersed in a perfect fluid. The control inputs are three forces f_1, f_2, f_3 applied at a point C located at a distance h from the center of mass G (see Fig. 1). We assume that these forces are aligned with the three principal axes of inertia and that \overrightarrow{CG} is aligned with the first one. The body's classical kinematic equations are given by

$$\dot{R} = R\hat{\omega}, \quad \dot{p} = Rv_G \quad (25)$$

with R the rotation matrix representing the body's attitude (w.r.t. an inertial frame), ω the angular velocity vector expressed in the body's frame, p the position vector of the center of mass, and v_G the velocity vector of G expressed in the body's frame. By denoting,

$$g := \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}, \quad \xi_G := \begin{pmatrix} \omega \\ v_G \end{pmatrix}$$

the equations (25) are equivalent to the following left-invariant system on the matrix Lie group $SE(3)$:

$$\dot{g} = g \hat{\xi}_G = X_G(g) \xi_G \quad (26)$$

with $X_{G,i}(g) := g \hat{e}_i$, ($i = 1, \dots, 6$). As for the dynamic equations, they are given by (see [4] for more details)

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + Mv_G \times v_G + h(0, f_3, -f_2)' \\ M\dot{v}_G = Mv_G \times \omega + f \end{cases} \quad (27)$$

with $J = \text{Diag}(j_1, j_2, j_3)$ the inertia matrix, and $M = \text{Diag}(m_1, m_2, m_3)$ the mass matrix.

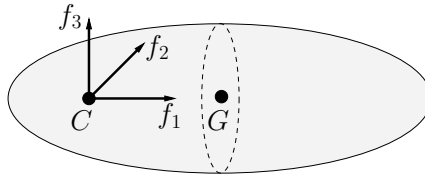


Figure 1: Rigid body in $SE(3)$ with three force controls

The following result, whose demonstration involves elementary calculations, asserts that the rigid body equations can be transformed into System (11) with ω_3 as a generator.

Lemma 3 Define² $v := v_G - c_3 e_1 \times \omega = (v_{G,1}, v_{G,2} + c_3 \omega_3, v_{G,3} - c_3 \omega_2)'$, with $c_3 := j_3 / (hm_2)$. Then, System (26)–(27) can be written as (11) (with A , B , and $P(\xi_2)$ specified in the appendix), by setting

$$\begin{cases} X_1 = X_{G,3} - c_3 X_{G,5}, X_2 = X_{G,1}, X_3 = X_{G,2} + c_3 X_{G,6}, X_i = X_{G,i} \ (i = 4, 5, 6) \\ \xi_1 = \omega_3, \xi_2 = (\omega_1, \omega_2, v_1, v_2, v_3)' \end{cases}$$

and

$$\begin{cases} u_1 = -\frac{h}{j_3} f_2 + \frac{j_{12}}{j_3} \omega_1 \omega_2 + \frac{m_{12}}{j_3} v_1 (v_2 - c_3 \omega_3) \\ u_2 = \left(f_1 + m_2 \omega_3 (v_2 - c_3 \omega_3) - m_3 \omega_2 (v_3 + c_3 \omega_2) \right) / f_3 \end{cases}$$

with $j_{ik} := j_i - j_k$ and $m_{ik} := m_i - m_k$. Furthermore, Assumption 1 is satisfied provided that

$$h^2 m_2 m_3 j_{23} - j_2 j_3 m_{23} \neq 0 \quad (28a)$$

$$h^2 m_1 m_2 + j_3 m_{12} \neq 0 \quad (28b)$$

The choice of ω_3 as a generator is not unique. For instance, by the system's symmetry, ω_2 is also a possible choice:

Lemma 4 Define $v := v_G - c_2 e_1 \times \omega$, with $c_2 := j_2 / (hm_3)$. Then, System (26)–(27) can be written as (11) (with A , B , and $P(\xi_2)$ specified in the appendix), by setting

$$\begin{cases} X_1 = X_{G,2} + c_2 X_{G,6}, X_2 = X_{G,1}, X_3 = X_{G,3} - c_2 X_{G,5}, X_i = X_{G,i} \ (i = 4, 5, 6) \\ \xi_1 = \omega_2, \xi_2 = (\omega_1, \omega_3, v_1, v_2, v_3)' \end{cases}$$

and

$$\begin{cases} u_1 = \frac{h}{j_2} f_3 + \frac{j_{31}}{j_2} \omega_1 \omega_3 + \frac{m_{31}}{j_2} v_1 (v_3 + c_2 \omega_2) \\ u_2 = \left(f_1 + m_2 \omega_3 (v_2 - c_2 \omega_3) - m_3 \omega_2 (v_3 + c_2 \omega_2) \right) / f_2 \end{cases}$$

with j_{ik} , and m_{ik} , defined as in Lemma 3. Furthermore, Assumption 1 is satisfied provided that

$$h^2 m_2 m_3 j_{23} - j_2 j_3 m_{23} \neq 0 \quad (29a)$$

$$h^2 m_1 m_3 + j_2 m_{13} \neq 0 \quad (29b)$$

It is not difficult to show that if (28a) is satisfied, then either (28b) or (29b) is satisfied. Therefore, by Proposition 2, (28a) is a sufficient condition for the Small Time Local Controllability of System (26)–(27). Note that this condition is weaker than the one stated in [4, Sec. III] where only “symmetric products” of order at most equal to two are used to derive it.

²Note that v is the velocity vector of the point P defined by $\overrightarrow{GP} = c_3 \vec{e}_1$.

Once System (26)–(27) is written as (11), the control design method of Section 4 applies directly, eventhough the calculation of some of the terms in the control expression (like the matrix e^{Ax} involved in the group law (15) and a function f transversal to the v.f. Y_1, Y_2, Y_3 , depending, in this case, on $6 - 3 = 3$ variables) is a little tedious. For the determination of a transverse function, one can use the expression [16, Eq. (10)] which gives f in terms of the group law (15). The analytic expression of e^{Ax} , for the matrix A associated with the generator ω_3 , is given in the appendix.

6 Choice of the generator: a case study

The previous example illustrates that there may be many ways to write System (1) as (11), in relation with the choice of the velocity variable used as a generator. While the ultimate boundedness of the tracking error is obtained independently of the chosen generator, other issues intervene when making this choice. For instance, we have observed in simulation that the asymptotical behavior of the controlled system's solutions can be very sensitive to this choice. In this respect, a meaningful criterion is the convergence, or non-convergence, to zero of the velocity variables when the reference trajectory is a fixed point. Let us examine this issue more closely. By Lemma 2, the control law (23) makes $\bar{\nu}$ tend to zero exponentially. On the zero-dynamics $\bar{\nu} = 0$ one deduces from (17)–(22) and (23) that, when $\xi_r = 0$,

$$\dot{\bar{g}} = X(\bar{g})\text{Ad}^X(h_1)\bar{\xi} \quad (30a)$$

$$\bar{\xi} = \begin{pmatrix} \xi_1^*(\bar{g}) \\ e^{Af_1}\xi_2^*(\bar{g}) + f_2 \end{pmatrix} \quad (30b)$$

$$\begin{pmatrix} u_2 \\ \alpha \end{pmatrix} = D(\alpha)^{-1} \left(e^{Af_1} \frac{\partial \xi_2^*(\bar{g})}{\partial \bar{g}} X(\bar{g})\text{Ad}^X(h_1) \begin{pmatrix} \xi_1^*(\bar{g}) \\ \xi_2 \end{pmatrix} - P(\xi_2) - \xi_1^*(\bar{g})A\xi_2 \right) \quad (30c)$$

with the argument α being omitted at several places to lighten the notation. A question of interest concerns the convergence, or non-convergence, of $\bar{\xi}$ to zero along the solutions to this system. Indeed, since $\bar{\xi}_1 = \xi_1 - \dot{f}_1(\alpha)$ and $\bar{\xi}_2 = \xi_2$, the convergence of $\bar{\xi}$ to zero is equivalent to the convergence of the velocity vector ξ to zero – a clearly desirable property in practice. This is a very challenging question in general. The answer depends, among other things, on the system, the chosen generator, and the feedback law ξ^* . In this section, these general considerations are made more precise by studying the special case of the second order chained system

$$\begin{cases} \ddot{x}_1 = u_1 \\ \ddot{x}_2 = u_2 \\ \ddot{x}_3 = u_1 x_2 \end{cases} \quad (31)$$

which can be used to model several underactuated mechanical systems evolving in a three-dimensional configuration space. This system can be written as (11) with $g = x = (x_1, x_2, x_3)'$, A, B , and P defined by (12) with $a = -1$. The group operation on \mathbb{R}^3 is defined by $xy = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + y_1 x_2)$. From there, a possibility consists in choosing \dot{x}_1 as the generator. This corresponds to setting $X_1(x) = (1, 0, x_2)'$, $X_2 = (0, 1, 0)'$, $X_3 = (0, 0, 1)'$,

and $\xi = (\dot{x}_1, \dot{x}_2, \dot{x}_3 - x_2\dot{x}_1)'$. Another possibility consists in choosing \dot{x}_2 as the generator. Then $X_1 = (0, 1, 0)'$, $X_2(x) = (1, 0, x_2)'$, $X_3 = (0, 0, 1)'$, and $\xi = (\dot{x}_2, \dot{x}_1, \dot{x}_3 - x_2\dot{x}_1)'$. Note that the respective roles of u_1 and u_2 then commute. We show below that, for a specific choice of ξ^* , this second solution yields a convergent zero-dynamics.

First, one easily verifies that a family of functions f transversal to the v.f. Y_1, Y_2 (i.e. such that the matrix $D(\alpha)$ is invertible for any α) is given in this case by

$$\begin{aligned} f(\alpha) &= \exp(\varepsilon_1 \sin(\alpha)Y_1 + \varepsilon_2 \cos(\alpha)Y_2) \\ &= (\varepsilon_1 \sin \alpha, \varepsilon_2 \cos \alpha, -\varepsilon_1 \varepsilon_2 \frac{\sin 2\alpha}{4})' \end{aligned} \quad (32)$$

with $\varepsilon_1, \varepsilon_2 > 0$. Furthermore, since $X_1 = (0, 1, 0)'$, it follows from (20) that $h_1 = (0, f_1, 0)'$. By calculating the matrix $\text{Ad}^X(h_1)$ or, more simply in this case, by differentiating the equality $\bar{g} = g - (0, f_1, 0)'$, Eq. (30a)–(30b) yield

$$\begin{cases} \dot{\bar{g}}_1 = \bar{\xi}_{2,1} &= \xi_{2,1}^* + f_{2,1} \\ \dot{\bar{g}}_2 = \bar{\xi}_1 &= \xi_1^* \\ \dot{\bar{g}}_3 = \bar{\xi}_{2,2} + g_2 \bar{\xi}_{2,1} &= \xi_{2,2}^* + \bar{g}_2 \xi_{2,1}^* + f_{2,2} + g_2 f_{2,1} \end{cases} \quad (33)$$

Now, in view of the v.f. $X_{1,2,3}$, the origin of the system $\dot{g} = X(g)\xi$ is clearly exponentially stabilized by the kinematic feedback $\xi^*(g) = -kg$, with $k > 0$. Let us thus make this choice for ξ^* . Then, by differentiating the equality $\dot{\bar{g}}_1 = \bar{\xi}_{2,1} = -k\bar{g}_1 + f_{2,1}$, one obtains

$$\dot{\bar{\xi}}_{2,1} = -k\bar{\xi}_{2,1} + \dot{f}_{2,1} \quad (34)$$

It follows from the second equation of (33) and the definition of ξ^* that \bar{g}_2 tends to zero exponentially and, subsequently, that $\bar{\xi}_1$ and $\xi_1^*(\bar{g})$ also tend to zero exponentially. From now on, let us set these three variables equal to zero. This corresponds to studying the system's zero-dynamics given by $(\bar{\nu} = 0, \bar{g}_2 = 0)$. By using (16) and (33), one deduces from (30c) that

$$\dot{\alpha} = \frac{2k}{\varepsilon_1 \varepsilon_2} \bar{\xi}_{2,2} \quad (35)$$

Since $\dot{\bar{\xi}}_{2,2} = -\xi_1 \bar{\xi}_{2,1}$, and $\xi_1 = \dot{f}_1$ on the zero-dynamics, one has $\dot{\bar{\xi}}_{2,2} = -\dot{f}_1 \bar{\xi}_{2,1}$ and, by (35),

$$\begin{aligned} \ddot{\alpha} &= -\frac{2k}{\varepsilon_1 \varepsilon_2} \dot{f}_1 \bar{\xi}_{2,1} \\ &= -\left(\frac{2k}{\varepsilon_2^2} f_{2,1}(\alpha) \bar{\xi}_{2,1} \right) \dot{\alpha} \end{aligned} \quad (36)$$

where the second equality follows from (32). We claim that (34) and (36) imply that $\bar{\xi}_{2,1}$ converges to zero. Let us make a proof by contradiction and assume that $\bar{\xi}_{2,1}$ does not converge to zero. Since $\bar{\xi}$ and $\dot{\alpha}$ are bounded along the system's solutions (because \bar{g} is

bounded), $\bar{\xi}_{2,1}$ is uniformly continuous. By multiplying both sides of Eq. (34) by $\bar{\xi}_{2,1}$, and by integrating the resulting equation, one obtains

$$\int_0^t \bar{\xi}_{2,1}(s) \dot{f}_{2,1}(\alpha(s)) ds \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty$$

By integrating this equation by part, and by using (34) again

$$I(t) := \int_0^t \bar{\xi}_{2,1}(s) f_{2,1}(\alpha(s)) ds \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty \quad (37)$$

By (36),

$$\dot{\alpha}(t) = \dot{\alpha}(0) \exp\left(-\frac{2k}{\varepsilon_2^2} I(t)\right)$$

and it follows from (37) that $\dot{\alpha}$ tends to zero. Therefore, in view of (34), $\bar{\xi}_{2,1}$ also tends to zero, i.e. a contradiction with our starting assumption. Now, since $\bar{\xi}_{2,1}$ tends to zero, it comes from (35) that $\dot{\alpha}$, $\bar{\xi}_{2,2}$, and ξ_1 also tend to zero. In other words, all velocities tend to zero. Although the above analysis is limited to the system's zero-dynamics, it is not very difficult to show that the same conclusion extends to the original system. This is done by incorporating exponentially vanishing terms in the right-hand sides of Eq. (34), (35), and (36).

Now, do the system's velocities also vanish asymptotically when choosing \dot{x}_1 as the generator? Let us first indicate that the system's zero dynamics associated with this choice yields a set of deceptively little more complicated equations (as the interested reader can verify) which, in fact, underly a much more complex situation which cannot be simply summarized by asserting either convergence, or non-convergence, of the system's velocities. However, all simulations that we have conducted tend to indicate that zero velocity is not stable in this case and, moreover, that non-convergence to zero is the generic situation. It is thus clear to us that, on the basis of the above elements, \dot{x}_2 is a "better" generator than \dot{x}_1 . However, it is also clear that this issue deserves to be studied further.

References

- [1] N.P.I. Aneke. *Control of underactuated mechanical systems*. PhD thesis, Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands, 2003.
- [2] A. Behal, D.M. Dawson, W.E. Dixon, and Y. Fang. Tracking and regulation control of an underactuated surface vessel with nonintegrable dynamics. *IEEE Trans. on Automatic Control*, 47(3):495–500, 2002.
- [3] R.W. Brockett. Asymptotic stability and feedback stabilization. In R.W. Brockett, R.S. Millman, and H.J. Sussmann, editors, *Differential Geometric Control Theory*. Birkhauser, 1983.

- [4] F. Bullo, N.H. Leonard, and A.D. Lewis. Controllability and motion algorithms for underactuated lagrangian systems on Lie groups. *IEEE Trans. on Automatic Control*, 45:1437–1454, 2000.
- [5] C.I. Byrnes and A. Isidori. On the attitude stabilization of rigid spacecraft. *Automatica*, 27:87–95, 1991.
- [6] J.-M. Coron. Stabilization in finite time of locally controllable systems by means of continuous time-varying feedback laws. *SIAM Journal on Control and Optimization*, 33:804–833, 1995.
- [7] J.-M. Coron and E.-Y. Kerai. Explicit feedbacks stabilizing the attitude of a rigid spacecraft with two control torques. *Automatica*, 32:669–677, 1996.
- [8] E. Kerai. Analysis of small time local controllability of the rigid body model. In *IFAC Conf. on System Structure and Control*, pages 645–650, 1995.
- [9] H.K. Khalil. *Nonlinear systems*. Prentice Hall, 2002.
- [10] K. Kobayashi. *Controllability analysis and control design of nonholonomic systems*. PhD thesis, Department of Mechanical Engineering, Kyoto University, Japan, 1999.
- [11] A.D. Lewis and R.M. Murray. Configuration controllability of simple mechanical control systems. *SIAM Journal on Control and Optimization*, 35:766–790, 1997.
- [12] D.A. Lizárraga. Obstructions to the existence of universal stabilizers for smooth control systems. *Mathematics of Control, Signals, and Systems*, 16:255–277, 2004.
- [13] D.A. Lizárraga, N.P.I. Aneke, and H. Nijmeijer. Robust point stabilization of underactuated mechanical systems via the extended chained form. *SIAM Journal on Control and Optimization*, 42:2172–2199, 2004.
- [14] D.A. Lizárraga and J.M. Sosa. Vertically transverse functions as an extension of the transverse function approach for second-order systems. In *IEEE Conf. on Decision and Control (CDC)*, pages 7290–7295, 2005.
- [15] P. Morin and C. Samson. Time-varying exponential stabilization of a rigid spacecraft with two control torques. *IEEE Trans. on Automatic Control*, 42:528–534, 1997.
- [16] P. Morin and C. Samson. Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Trans. on Automatic Control*, 48:1496–1508, 2003.
- [17] P. Morin and C. Samson. Control of underactuated mechanical systems by the transverse function approach. In *IEEE Conf. on Decision and Control (CDC)*, pages 7508–7513, 2005.

- [18] P. Morin, C. Samson, J.-B. Pomet, and Z.-P. Jiang. Time-varying feedback stabilization of the attitude of a rigid spacecraft with two controls. *Systems & Control Letters*, 25:375–385, 1995.
- [19] H.J. Sussmann. A general theorem on local controllability. *SIAM Journal on Control and Optimization*, 25:158–194, 1987.

Appendix

Proof of Prop. 2:

System (11) can be written in the classical control affine form

$$\dot{y} = Y_0(y) + \sum_{i=1}^m u_i Y_i(y) \quad (38)$$

with

$$y = \begin{pmatrix} g \\ \xi_1 \\ \xi_2 \end{pmatrix}, \quad Y_0(y) = \begin{pmatrix} X(g)\xi \\ 0 \\ \xi_1 A \xi_2 + P(\xi_2) \end{pmatrix}, \quad Y_1(y) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$Y_i(y) = \begin{pmatrix} 0 \\ 0 \\ B e_i \end{pmatrix} \quad (i = 2, \dots, m)$$

Let $\langle Y_i : Y_j \rangle := [Y_i, [Y_0, Y_j]]$ (see [11] for more details on the special role played by this type of Lie bracket), and define inductively $\text{ads}_{Y_1}^0(Y_i) := Y_i$ and $\text{ads}_{Y_1}^k(Y_i) := \langle Y_1 : \text{ads}_{Y_1}^{k-1}(Y_i) \rangle$ for $k > 0$. Then, one easily verifies from the above equations that

$$\text{ads}_{Y_1}^k(Y_i) = (-1)^k \begin{pmatrix} 0 \\ 0 \\ A^k B e_i \end{pmatrix} \quad (i = 2, \dots, m)$$

It follows from this equality and from Assumption 1 that the vectors obtained by evaluating the vector fields

$$Y_1, \text{ads}_{Y_1}^k(Y_i), [Y_0, Y_1], [Y_0, \text{ads}_{Y_1}^k(Y_i)] \quad (39)$$

at any point y , with $i = 1, \dots, m$ and $k = 0, \dots, n-2$, span the tangent space (i.e. \mathbb{R}^n) at this point, so that System (38) satisfies the Lie Algebra Rank Condition at any point. Furthermore, all the Lie brackets associated with these v.f. are “good”³ in the sense of [19]. In order to prove that System (38) is STLC at $y = (g_0, 0, 0)$ it is sufficient, according to [19, Sec. 7.3], to find positive weights (w_0, w_1, \dots, w_m) with $w_0 \leq w_i$ ($i = 1, \dots, m$) such that

³To be rigorous, one should introduce here the notion of formal brackets; this distinction is here omitted due to space limitations.

all “bad” brackets (i.e. containing an odd number of Y_0 and an even number of each Y_i ($i = 1, \dots, m$)) can be expressed, when they are evaluated at $(g_0, 0, 0)$, as linear combinations of lower-degree brackets, where the degree of a bracket is defined by $\deg(Y_i) = w_i$ and, recursively, by $\deg([B_1, B_2]) = \deg(B_1) + \deg(B_2)$. We claim that this property is satisfied with $(w_0, w_1, \dots, w_m) = (1, 1, 2n, \dots, 2n)$. One only has to check that

- i) Each bad bracket containing one of the v.f. Y_2, \dots, Y_m is of degree strictly larger than the degrees of the v.f. in (39),
- ii) Each bad bracket containing none of the v.f. Y_2, \dots, Y_m is equal to zero when evaluated at $y = (g, 0, 0)$.

The proof of i) is straightforward, and the proof of ii) readily follows from the following property:

Property: Let $Y := [Y_{i_1}, [\dots, [Y_{i_{k-1}}, Y_{i_k}] \dots]]$ with each $i_j \in \{0, 1\}$ and $k > 1$. Let us denote by $n_0(Y)$ (resp. $n_1(Y)$) the number of integers i_j equal to 0 (resp. equal to 1). Consider the following decomposition of Y : $Y = Y^g \frac{\partial}{\partial g} + Y^{\xi_1} \frac{\partial}{\partial \xi_1} + Y^{\xi_2} \frac{\partial}{\partial \xi_2}$. Then,

- 1. $Y^{\xi_1}(y) = 0$,
- 2. $Y^{\xi_2}(y)$ does not depend on g , and vanishes for $\xi_2 = 0$,
- 3. $Y^g(y)$ (resp. $Y^{\xi_2}(y)$) is a vector of homogeneous polynomials in the components of ξ of degree $n_0(Y) - n_1(Y)$ (resp. of degree $n_0(Y) - n_1(Y) + 1$).

The proof of this property is easily obtained by induction on the length k of the Lie bracket Y , starting from $k = 2$.

Proof of Prop. 3:

In view of Prop. 2, it is sufficient to prove that if System (1) is STLC, then it can be written as (11) with Assumption 1 being satisfied. When $m = 1$, it can be shown, as in [8, Sec. 3], that System (1) is not STLC. Therefore nothing else needs to be proved in this case. When $m = 2$, let P denote any matrix such that $Pb_1 = e_1$ and $Pb_2 = e_2$. Such a matrix clearly exists since b_1 and b_2 are, by assumption, independent vectors. By making the linear change of variable $\xi \mapsto \bar{\xi} = P\xi$, and by a change of control input $u \mapsto \bar{u}$, one transforms System (1) into a system

$$\begin{cases} \dot{g} &= \bar{X}(g)\bar{\xi} \\ \dot{\bar{\xi}}_1 &= \bar{u}_1 \\ \dot{\bar{\xi}}_{2,1} &= \bar{u}_2 \\ \dot{\bar{\xi}}_{2,2} &= \bar{Q}_3(\bar{\xi}) \end{cases} \quad (40)$$

with \bar{X}_i ($i = 1, 2, 3$) being left-invariant on G , and \bar{Q}_3 a quadratic form in the components of $\bar{\xi}$. This quadratic form can be decomposed as

$$\bar{Q}_3(\bar{\xi}) = \bar{Q}_{3,1}(\bar{\xi}_1, \bar{\xi}_{2,1}) + \bar{\xi}_{2,2}L(\bar{\xi})$$

with $\bar{Q}_{3,1}$ a quadratic form in $\bar{\xi}_1, \bar{\xi}_{2,1}$, and L a linear form in $\bar{\xi}$. We claim that $\bar{Q}_{3,1}$ can be neither non-positive nor non-negative. Indeed, suppose for instance that $\bar{Q}_{3,1}$ is non-negative. Then $\dot{\bar{\xi}}_{2,2}(t)$ is non-negative when $\bar{\xi}_{2,2}(t) = 0$. By application of the comparison lemma [9, Lem. 3.4], it results that $\bar{\xi}_{2,2}(t)$ can never be negative if $\bar{\xi}_{2,2}(0) = 0$, thus yielding a contradiction with the assumption of Small Time Local Controllability. Since $\bar{Q}_{3,1}$ is neither non-positive nor non-negative, it has to be of the form

$$\bar{Q}_{3,1} = a_1(\bar{\xi}_1 + a_2\bar{\xi}_{2,1})(\bar{\xi}_{2,1} + a_3\bar{\xi}_1) \quad (41)$$

with $a_1 \neq 0$ and $a_2a_3 \neq 1$. By the new linear change of coordinates $\bar{\xi} \mapsto \bar{\bar{\xi}} := (\bar{\xi}_1 + a_2\bar{\xi}_{2,1}, \bar{\xi}_{2,1} + a_3\bar{\xi}_1, \bar{\xi}_{2,2})'$ and the change of control inputs $\bar{u} \mapsto \bar{\bar{u}} := (\bar{u}_1 + a_2\bar{u}_2, \bar{u}_2 + a_3\bar{u}_1)'$, System (40) is transformed into

$$\begin{cases} \dot{\bar{\bar{\xi}}} &= \bar{\bar{X}}(g)\bar{\bar{\xi}} \\ \dot{\bar{\bar{\xi}}}_1 &= \bar{\bar{u}}_1 \\ \dot{\bar{\bar{\xi}}}_2 &= \bar{\bar{\xi}}_1 A \bar{\bar{\xi}}_2 + B u_2 + P(\bar{\bar{\xi}}_2) \end{cases}$$

with

$$\bar{\bar{X}}(g) = \frac{1}{1 - a_2a_3} \bar{X}(g) \begin{pmatrix} 1 & -a_2 & 0 \\ -a_3 & 1 & 0 \\ 0 & 0 & 1 - a_2a_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 \\ a_1 & l_1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P(\bar{\bar{\xi}}_2) = \begin{pmatrix} 0 \\ \bar{\bar{\xi}}_{2,2}(l_2\bar{\bar{\xi}}_{2,1} + l_3\bar{\bar{\xi}}_{2,2}) \end{pmatrix}$$

for some numbers l_1, l_2 , and l_3 . Moreover, since $a_1 \neq 0$, the pair (A, B) is controllable.

Expressions of A , B , and $P(\xi_2)$ with ω_3 as the generator:

$$A = \begin{pmatrix} 0 & \frac{j_{23} - c_3^2 m_{23}}{j_1} & 0 & 0 & -\frac{c_3 m_{23}}{j_1} \\ \frac{j_{31}}{j_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1}{m_2} - \frac{c_3^2 m_{12}}{j_3} & 0 & 0 \\ \frac{c_3 m_{12}}{m_3} - \frac{c_3 j_{31}}{j_2} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{h}{j_2} \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_3} - \frac{c_3 h}{j_2} \end{pmatrix}$$

and

$$P(\xi_2) = \begin{pmatrix} \frac{m_{23}}{j_1} v_2 (v_3 + c_3 \omega_2) \\ \frac{m_{31}}{j_2} v_1 (v_3 + c_3 \omega_2) \\ 0 \\ \frac{m_3}{m_2} \omega_1 (v_3 + c_3 \omega_2) + \frac{c_3}{j_3} (j_{12} \omega_1 \omega_2 + m_{12} v_1 v_2) \\ \frac{m_1}{m_3} \omega_2 v_1 - \frac{m_2}{m_3} \omega_1 v_2 - \frac{c_3 m_{31}}{j_2} v_1 (v_3 + c_3 \omega_2) \end{pmatrix}$$

Expressions of A , B , and $P(\xi_2)$ with ω_2 as the generator:

$$A = \begin{pmatrix} 0 & \frac{j_{23}-c_2^2 m_{23}}{j_1} & 0 & \frac{c_2 m_{23}}{j_1} & 0 \\ \frac{j_{12}}{j_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{c_2 m_3}{m_2} + \frac{c_2 j_{12}}{j_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_1}{m_3} - \frac{c_2^2 m_{31}}{j_2} & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{h}{j_3} \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} - \frac{c_2 h}{j_3} \\ 0 & 0 \end{pmatrix}$$

and

$$P(\xi_2) = \begin{pmatrix} \frac{m_{23}}{j_1} v_3 (v_2 - c_2 \omega_3) \\ \frac{m_{12}}{j_3} v_1 (v_2 - c_2 \omega_3) \\ 0 \\ \frac{m_3}{m_2} \omega_1 v_3 - \frac{m_1}{m_2} \omega_3 v_1 + \frac{c_2 m_{12}}{j_3} v_1 (v_2 - c_2 \omega_3) \\ -\frac{m_2}{m_3} \omega_1 (v_2 - c_2 \omega_3) - \frac{c_2}{j_2} (j_{31} \omega_1 \omega_3 + m_{31} v_1 v_3) \end{pmatrix}$$

Expressions of e^{Ax} , with ω_3 as the generator:

Let $\Delta := \sqrt{a_{12}a_{21} + a_{15}a_{51}}$ with a_{ij} the (i, j) element of A , and note that $\Delta \neq 0$ if Condition (28) is satisfied. Let $ch(x) := \cosh(\Delta x)$, $sh(x) := \sinh(\Delta x)$. Then,

$$e^{Ax} = \begin{pmatrix} ch(x) & \frac{a_{12} sh(x)}{\Delta} & 0 & 0 & \frac{a_{15} sh(x)}{\Delta} \\ \frac{a_{21} sh(x)}{\Delta} & \frac{a_{15} a_{51} + a_{12} a_{21} ch(x)}{\Delta^2} & 0 & 0 & \frac{a_{15} a_{21} (ch(x) - 1)}{\Delta^2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_{43} x & 1 & 0 \\ \frac{a_{51} sh(x)}{\Delta} & \frac{a_{12} a_{51} (ch(x) - 1)}{\Delta^2} & 0 & 0 & \frac{a_{12} a_{21} + a_{15} a_{51} ch(x)}{\Delta^2} \end{pmatrix}$$



Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)

Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
<http://www.inria.fr>
ISSN 0249-6399